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ON A GEOMETRIC PROPERTY OF PERFECT GRAPHS

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Let G be a graph, $VP(G)$ its vertex packing polytope and let $A(G)$ be obtained by reflecting $VP(G)$ in all Cartersian coordinates. Denoting by $A^*(G)$ the set obtained similarly from the fractional vertex packing polytope, we prove that the segment connecting any two non-antipodal vertices of $A(G)$ is contained in the surface of $A(G)$ and that G is perfect if and only if $A^*(G)$ has a similar property.

1. Introduction

A graph G is called γ-perfect if and only if for every one of its induced subgraphs G_0, the chromatic number of G_0 equals the maximum number of vertices forming a clique in G_0; a graph G is α-perfect if and only if for every one of its induced subgraphs G_0, the minimum number of cliques covering G_0 is equal to the size of a largest independent (stable) set in G_0. These concepts were introduced by C. Berge in the early nineteen-sixties to formulate two conjectures. One of them stating that a graph is γ-perfect if and only if it is α-perfect was proved in 1971 by Lovász [4, 5] and became known as the Perfect Graph Theorem, whereas the second, called the Strong Perfect Graph Conjecture, is still unsettled.

A further result [6, p. 86] (essentially used in this paper) that is strongly related to perfect (i.e., γ or, equivalently α-perfect graphs) is due to Chvátal and Fulkerson [1, 2, 3]. It can be restated as follows. Let G be a finite undirected graph on n vertices; let A be the $m\times n$ clique-vertex incidence matrix of G, and B be the $r\times n$ clique-vertex incidence matrix of $\bar{G}$, the complement of G. Then the following conditions are equivalent (below e_r and e_m denote the vectors in R^r and R^m, respectively, with all their components equal to one)

(i) G is perfect,
(ii) $P(A)=\{x\in R^n\colon\ x\geqq 0,\ Ax\leqq e_m\}$ has only integer vertices,
(iii) $Q(B)=\{y\in R^n\colon\ y\geqq 0,\ By\leqq e_r\}$ has only integer vertices.

In this paper, a new condition characterizing perfect graphs is stated and proved. Namely, let G be a graph, $VP(G)$ its vertex packing polytope and let $A(G)$

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be obtained by reflecting $VP(G)$ in all Cartesian coordinates. Let, moreover, $A^*(G)$ be obtained similarly from the fractional vertex packing polytope. It is proved that the segment connecting any two non-antipodal vertices of $A(G)$ lies in the surface of $A(G)$ and that G is perfect if and only if $A^*(G)$ has similar property.

2. Basic notation

Let $G=(V,\Gamma)$ be a graph with n vertices. By A we shall denote the $m\times n$ clique matrix of G, and by B the $r\times n$ clique matrix of $\bar{G}$. We call

$$Q(B) = \{y\in R^n\colon\ y \geqq 0, \quad By \leqq e_r\} \tag{1}$$

the fractional vertex packing polytope of G; above e_r is the vector with all his r components equal to one.

By $\mathscr{E}$ we shall denote the family of all maximal cliques in G and by $\bar{\mathscr{E}}$ the family of all maximal stable sets in G. Besides, for every $W\subset V$, let x^W mean the incidence vector of W, i.e. $x_v^W=1$ if $v\in W$ and $x_v^W=0$ if $v\notin W$.

Therefore

$$VP(G) = \operatorname{conv}\{x^W\in R^n\colon\ W\in\bar{\mathscr{E}}\} \tag{2}$$

is the vertex packing polytope of G. Reflecting this set in all coordinate planes we obtain the set

$$A(G) = \operatorname{conv}\{x\in R^n\colon\ |x|\in VP(G)\}, \tag{3}$$

where $|x|=(|x_1|, \ldots, |x_n|)$. Let $A^*(G)$ be obtained similarly from the fractional vertex packing polytope of G, i.e.,

$$A^*(G) = \{x\in R^n\colon\ |x|\in Q(B)\}. \tag{4}$$

3. Main results

Theorem 1. *For any undirected graph G, the segment connecting any two non-antipodal vertices of $A(G)$ lies in the surface of $A(G)$.*

Proof. Let x^1 and x^2 be any pair of non-antipodal extreme points of $A(G)$. Assume that, for a certain number λ, $0<\lambda<1$, the point $\bar{x}=\lambda x^1+(1-\lambda)x^2\in \operatorname{int} A(G)$, the interior of $A(G)$. Then also $\bar{x}\in\operatorname{int} A^*(G)$, which means that, for some $\varepsilon>0$,

$$B|\bar{x}| < (1-\varepsilon)e_r. \tag{5}$$

Let $V_1=\{v\in V\colon\ x_v^1\neq 0\}$ and $V_2=\{v\in V\colon\ x_v^2\neq 0\}$. If $V_1=V_2$, then, for some index $v\in V$, we have $x_v^1=x_v^2\in\{+1, -1\}$ and consequently $|\bar{x}_v|=1$, which obviously contradicts (5).

In the opposite case, we get nodes $v_1\in V_1$ and $v_2'\in V$ such that, for some clique $E\in\bar{\mathscr{E}}$, $\{v_1, v_2'\}\subset E$. Then $x^E\cdot|\bar{x}|=\lambda|x_{v_1}^1|+(1-\lambda)|x_{v_2}^2|=1$, which again contradicts (5). ∎

Now we can state and prove our main result.

Theorem 2. *Let G be a graph on n vertices. Then the following conditions are equivalent:*
(i) *G is perfect,*
(ii) $A(G)=A^*(G)$,
(iii) *the segment connecting any two non-antipodal vertices of* $A^*(G)$ *lies in the surface of* $A^*(G)$.

Proof. We shall prove the equivalences (i)⇔(ii) and (ii)⇔(iii). To prove the first one, observe that condition (ii) is equivalent to the equality

$$VP(G) = Q(B), \tag{6}$$

which, by virtue of the well-known characterization of perfect graphs due to V. Chvátal and D. Fulkerson [6, p. 86], is equivalent to condition (i).

Assume now that condition (ii) does not hold. Then there exists a vertex $x \in A^*(G)$ with $|x_i|<1$ for some $i \in V$. Putting $y=(y_v)$, $v \in V$,

$$y_v = x_v, \quad v = i, \quad \text{and} \quad y_i = -x_i, \quad v \neq i, \tag{7}$$

and $z=\frac{1}{2}(x+y)$ we infer that the point z does not lie in the surface of $A^*(G)$. In this way we have proved (iii)⇒(ii). The reverse implication follows easily from Theorem 1. ∎

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